# Student Workbook

Solving Systems with Determinants



Name: ______________________________ Date: ________________ Class: _________

**ESSENTIAL QUESTION**

How does Cramer's Rule use determinants to express the solution to a linear system as a ratio of determinants, and when is this method most useful?

## Lesson Overview

Cramer's Rule gives explicit formulas for each variable in a linear system using determinants: $x_i = D_i/D$, where D = det(A) and $D_i$ is det(A) with column i replaced by the constant vector b. It only works when D ≠ 0.

### 2×2: det = ad−bc | 3×3: Cofactor Expansion | $x_i = D_i / D$

| | |
|---|---|
| **2×2 Det.** | det([[a,b],[c,d]]) = ad − bc (main diagonal product minus anti-diagonal product). |
| **3×3 Det.** | det([[a,b,c],[d,e,f],[g,h,i]]) = a(ei−fh) − b(di−fg) + c(dh−eg), via cofactor expansion along row 1. |
| **Cramer's Rule** | D = det(A); $D_x$, $D_y$, $D_z$ replace the corresponding column of A with b. $x=D_x/D$, $y=D_y/D$, $z=D_z/D$ — only when D≠0. |
| **Method Fails** | If D=0, the system is either inconsistent or dependent — no unique solution exists. |
| **Advantage** | Gives a formula for each variable independently — useful for theoretical work and when only one variable is needed. |

## Worked Examples

**EXAMPLE 1**

**Solve using Cramer's Rule: 3x + y = 7, 5x + 2y = 12.**

- D = det([[3,1],[5,2]]) = 6−5 = 1
- $D_x$ = det([[7,1],[12,2]]) = 14−12 = 2. $D_y$ = det([[3,7],[5,12]]) = 36−35 = 1
- x = $D_x/D$ = 2. y = $D_y/D$ = 1

**Answer: (2, 1)**

**EXAMPLE 2**

**Solve using Cramer's Rule: 2x − 3y = 4, x + 5y = −1.**

- D = det([[2,−3],[1,5]]) = 10+3 = 13
- $D_x$ = det([[4,−3],[−1,5]]) = 20−3 = 17. $D_y$ = det([[2,4],[1,−1]]) = −2−4 = −6
- x = 17/13. y = −6/13

**Answer: x = 17/13, y = −6/13**

**EXAMPLE 3**

**Evaluate the 3×3 determinant: det([[1,2,3],[4,5,6],[7,8,9]]).**

- Expand along row 1: = 1·det([[5,6],[8,9]]) − 2·det([[4,6],[7,9]]) + 3·det([[4,5],[7,8]])
- = 1·(45−48) − 2·(36−42) + 3·(32−35) = −3+12−9

**Answer: 0**

**EXAMPLE 4**

**Solve using Cramer's Rule: x+y+z=6, 2x−y+z=3, x+2y−z=2.**

- D = det([[1,1,1],[2,−1,1],[1,2,−1]]) = −1+3+5 = 7
- $D_x$ = det([[6,1,1],[3,−1,1],[2,2,−1]]) = −6+5+8 = 7 → x=1
- $D_y$ = det([[1,6,1],[2,3,1],[1,2,−1]]) = −5+18+1 = 14 → y=2. z=6−1−2=3

**Answer: (1, 2, 3)**

**EXAMPLE 5**

**Determine the solution type: 4x − 2y = 6, 2x − y = 3.**

- D = det([[4,−2],[2,−1]]) = −4+4 = 0
- $D_x$ = det([[6,−2],[3,−1]]) = −6+6 = 0. $D_y$ = det([[4,6],[2,3]]) = 12−12 = 0
- D = 0 and $D_x$ = $D_y$ = 0 → dependent system

**Answer: Infinitely many solutions (dependent system)**

## Guided Practice

Work through each problem below. Use the hint if you get stuck — write your final answer on the last line.

**1** Solve using Cramer's Rule: 4x + 3y = 10, 2x − y = 4.

*Hint: D = det([[4,3],[2,−1]]); $D_x$ = det([[10,3],[4,−1]]); $D_y$ = det([[4,10],[2,4]]).*

**2** Evaluate: det([[3,−1],[2,4]]).

*Hint: Use the formula ad − bc.*

**3** Evaluate the 3×3 determinant: det([[2,1,0],[3,−1,2],[1,4,−1]]).

*Hint: Expand along row 1.*

**4** Solve using Cramer's Rule: $2x + y - z = 3$, $x - y + 2z = 1$, $3x + 2y + z = 8$.

*Hint: Compute $D$, $D_x$, $D_y$, $D_z$ using cofactor expansion.*

**5** A system has $D = 0$ and $D_x = 5$. What can you conclude?

*Hint: What does $D = 0$ mean for Cramer's Rule?*

## Key Vocabulary

| | |
|---|---|
| **Determinant** | A scalar value computed from a square matrix; $\det(A) \neq 0$ iff A is invertible. |
| **2×2 Determinant** | det([[a,b],[c,d]]) = ad − bc. |
| **3×3 Determinant** | Computed by cofactor expansion along any row or column using 2×2 minors. |
| **Cramer's Rule** | $x_i = D_i/D$ where $D=\det(A)$ and $D_i$ is det(A) with column i replaced by b. |
| **Cofactor Expansion** | Expanding a determinant along a row or column using 2×2 minors with alternating signs (+−+). |
| **Minor** | The determinant of the submatrix obtained by deleting one row and one column. |

## Quick Check Quiz

Circle the letter of the correct answer for each question.

**1** **det([[3,2],[1,4]]) =** Multiple Choice

A 10
B 14
C 7
D −2

**2** **In Cramer's Rule, $D_x$ is formed by:** Multiple Choice

A Replacing row 1 with the constants
B Replacing column 1 with the constants
C Replacing the diagonal with the constants
D Transposing the matrix

**3** **If $D = 0$ and $D_x \neq 0$, the system is:** Multiple Choice

A Consistent with a unique solution
B Consistent and dependent
C Inconsistent
D Underdetermined

**4** **Solve: $2x + y = 5$, $x - y = 1$ using Cramer's Rule. $x =$** Multiple Choice

- **A** 1
- **B** 2
- **C** 3
- **D** 4

**5** **The 3×3 determinant is computed using:** Multiple Choice

- **A** The 2×2 formula applied three times
- **B** Cofactor expansion along a row or column
- **C** Adding all entries
- **D** Multiplying the diagonal entries

## Common Mistakes to Avoid

✗ Replacing the wrong column when forming $D_x$, $D_y$, $D_z$.

✓ For $x_i$, replace column i (the column of coefficients for variable $x_i$) with the constant vector b.

✗ Forgetting the alternating signs in cofactor expansion: +, −, +.

✓ The cofactor expansion has alternating signs: $a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$. The sign pattern is (+−+/−+−/+−+).

✗ Using Cramer's Rule when D = 0.

✓ Cramer's Rule only works when $D \neq 0$. If $D = 0$, the system has no unique solution — use elimination to determine if it's inconsistent or dependent.

✗ Computing det([[a,b],[c,d]]) as ab − cd.

✓ The 2×2 determinant is $ad - bc$ (main diagonal product minus anti-diagonal product), not $ab - cd$.

## Math Tips

★ Cramer's Rule is elegant for 2×2 systems and for theoretical work. For 3×3 and larger, Gaussian elimination is usually faster computationally.

★ When expanding a 3×3 determinant, choose the row or column with the most zeros — it minimizes computation.

★ The sign pattern for cofactor expansion: top-left is +, then alternates. For a 3×3: [[+,−,+],[−,+,−],[+,−,+]].

★ Cramer's Rule is useful when you only need ONE variable — compute just that $D_i$ and $D$, not all three.

★ If $D = 0$ and all $D_i = 0$, the system is dependent. If $D = 0$ and any $D_i \neq 0$, the system is inconsistent.